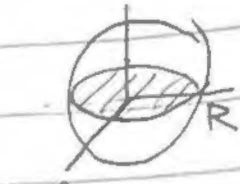
Double Integrals w/ spheres

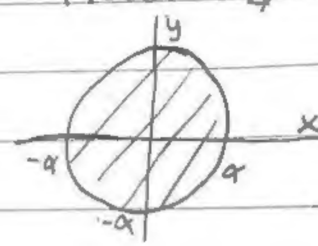
Last time: Double Integrals
This time: Double Integrals
Q: Suppose a sphere has radius α and is centered at the origin. What is its volume?



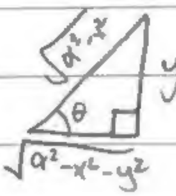
$x^2+y^2+z^2=\alpha^2$

Sol: we seek volume via double integral
For height function, $z^2=\alpha^2-x^2-y^2$
so upper hemisphere has height $z=\sqrt{\alpha^2-x^2-y^2}$
lower hemisphere has height $z=-\sqrt{\alpha^2-x^2-y^2}$
$\therefore$ Total height is "(upper hemisphere) - (lower hemisphere)"
i.e. $h(x,y)=\sqrt{\alpha^2-x^2-y^2}-\left(-\sqrt{\alpha^2-x^2-y^2}\right)$
$=2\sqrt{\alpha^2-x^2-y^2}$
To parameterize R_α, we note $R_\alpha=\{(x,y): x^2+y^2\le\alpha^2\}$
$=\{(x,y): -\alpha\le x\le\alpha,\ -\sqrt{\alpha^2-x^2}\le y\le\sqrt{\alpha^2-x^2}\}$

Pic. of R_α



$\therefore \text{Vol}(S)=\iint_R h(x,y)\,dA$
$=\int_{x=-\alpha}^{\alpha}\int_{y=-\sqrt{\alpha^2-x^2}}^{\sqrt{\alpha^2-x^2}} 2\sqrt{\alpha^2-x^2-y^2}\,dy\,dx$



Inner Int: $\int 2\sqrt{(\alpha^2-x^2)-y^2}\,dy$
$=2\int\sqrt{\alpha^2-x^2}\cos\theta\sqrt{\alpha^2-x^2}\cos\theta\,d\theta$

$y=\sqrt{\alpha^2-x^2}\sin\theta$
$dy=\sqrt{\alpha^2-x^2}\cos\theta\,d\theta$

$=2(\alpha^2-x^2)\int\cos^2(\theta)\,d\theta$
$=(\alpha^2-x^2)\int(1+\cos(2\theta))\,d\theta$

$\sin(2\theta)=2\sin\theta\cos\theta$

$=(\alpha^2-x^2)\left(\theta+\frac{1}{2}\sin(2\theta)\right)+C$
$=(\alpha^2-x^2)(\theta+\sin\theta\cos\theta)+C$
$=(\alpha^2-x^2)\left(\arcsin\left(\frac{y}{\sqrt{\alpha^2-x^2}}\right)+\left(\frac{y}{\sqrt{\alpha^2-x^2}}\right)\cdot\left(\frac{\sqrt{\alpha^2-x^2-y^2}}{\sqrt{\alpha^2-x^2}}\right)\right)+C$

$=(\alpha^2-x^2)\left(\arcsin\left(\frac{y}{\sqrt{\alpha^2-x^2}}\right)\right)+y\sqrt{\alpha^2-x^2-y^2}+C$

$\longrightarrow$

Sol: $\rightarrow$

$\therefore$ evaluating we obtain

$$\int_{y=\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\sqrt{a^2-x^2-y^2}\,dy = \left[(a^2-x^2)\arcsin\left(\frac{y}{\sqrt{a^2-x^2}}\right)+y\sqrt{a^2-x^2-y^2}\right]_{y=\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}$$

$$=\left(\left((a^2-x^2)\arcsin(1)+\sqrt{a^2-x^2}\sqrt{0}\right)-\left((a^2-x^2)\arcsin(-1)-\cancel{\sqrt{a^2-x^2}\sqrt{0}}\right)\right)$$

$$=(a^2-x^2)\left(\arcsin(1)-\arcsin(-1)\right)=(a^2-x^2)\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)$$

$$=\pi(a^2-x^2)$$

Outer Integral: $\int_{x=-a}^{a}\pi(a^2-x^2) = \pi\int_{x=-a}^{a}(a^2-x^2)\,dx$

$$=\pi\left[a^2x-\frac{1}{3}x^3\right]_{x=-a}^{a} = \pi\left(\left(a^3-\frac{1}{3}a^3\right)-\left(-a^3+\frac{1}{3}a^3\right)\right)$$

$$=\pi\left(2a^3-\frac{2}{3}a^3\right)=\frac{4}{3}\pi a^3$$

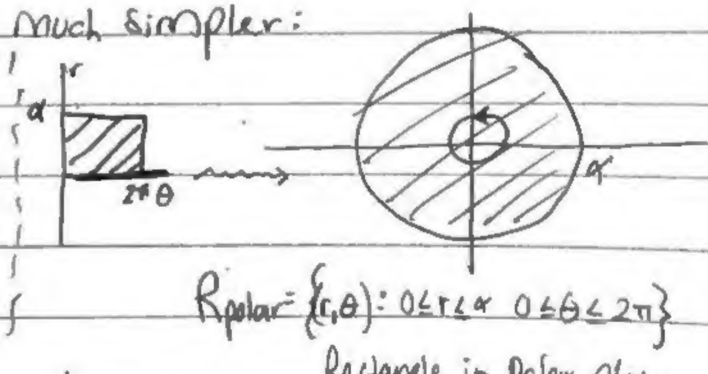
Hence the $V(S_a)=\frac{4}{3}\pi a^3$ is the Volume of a sphere of radius $a>0$ $\square$

NB: That was Computationally Complicated. If we understand how to use polar coordinates for integration, both the radius and height function are much simpler:

height function $h(x,y)=\sqrt{a^2-x^2-y^2}$

So, $h(r\cos\theta, r\sin\theta)=\sqrt[2]{a^2-r^2}$

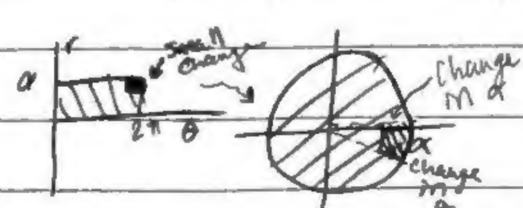
NEED: Understand dA polar and how it relates to dA cartesian.



$R_{polar}=\{(r,\theta): 0\le r\le a \quad 0\le\theta\le 2\pi\}$

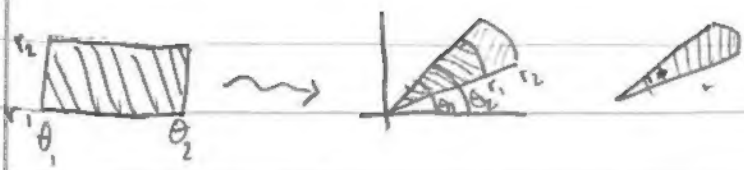
Rectangle in Polar plane

To understand dApolar, consider a small



polar rectangle. In the Cartesian plane this corresponds to a Circular sector.

$\longrightarrow$



$\frac{1}{2}r^2\theta$ = area of a full circular sheet

Area of a (small) circular sector is:

$$\left(\frac{1}{2}r_2^2(\theta_2-\theta_1)\right)-\left(\frac{1}{2}r_1^2(\theta_2-\theta_1)\right)=\frac{1}{2}(\theta_2-\theta_1)(r_2^2-r_1^2)=\frac{1}{2}(r_2+r_1)(r_2+r_1)(\theta_2-\theta_1)$$

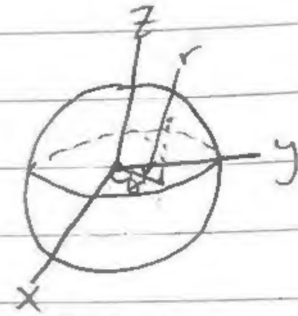
$$\therefore\ \Delta A_{cart}=\frac{1}{2}(r_1+r_2)\Delta r\,\Delta\theta=\frac{1}{2}(r_1+r_2)\,\Delta A_{polar}$$

Now limiting as $\Delta A\to 0$ (i.e. $\Delta\theta\to 0$ and $\Delta r\to 0$).

we see $\frac{1}{2}(r_1+r_2)\longrightarrow\frac{1}{2}2r^*=r^*$

Hence in the limit we obtain $\underset{\substack{dy\,dx\\ \text{or}\\ dx\,dy}}{dA_{cart}}=r\,\underset{\substack{dr\,d\theta\\ \text{or}\\ d\theta\,dr}}{dA_{polar}}$

Volume of Sphere, a. (polar coordinates)



Sol: In polar Coords (r,θ,z) space, Integral becomes

$$Vol(S_a)=\iint_{R_{cart}}h(x,y)\,dA_{cart}$$

$$=\iint_{R_{polar}}h(r\cos\theta,r\sin\theta)\,r\,dA_{polar}=\int_{\theta=0}^{2\pi}\int_{r=0}^{a}2\sqrt{a^2-r^2}\,r\,dr\,d\theta$$

$z=\sqrt{a^2-x^2-y^2}$
$=\sqrt{a^2-r^2}$

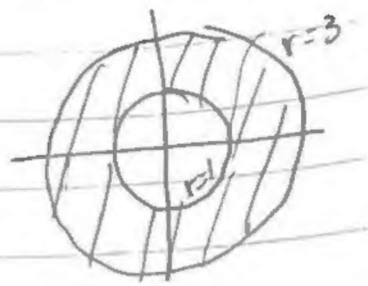
$R_{polar}=[0,2\pi]\times[0,a]$
(θ,r)

Inner Integral: $\int_{r=0}^{a}2r\sqrt{a-r^2}\,dr$ $\quad u=a^2-r^2$, $du=-2r\,dr$

$$=-\int_{r=0}^{a}u^{1/2}\,du=-\left[\frac{2}{3}u^{3/2}\right]_{r=0}^{a}$$

$$=-\frac{2}{3}\left[(a^2-r^2)^{3/2}\right]_{r=0}^{a}=\frac{-2}{3}\left((0)^{3/2}-(a^2)^{3/2}\right)=\frac{-2}{3}(-a^3)=\frac{2}{3}a^3$$

Outer: $\int_{\theta=0}^{2\pi}\frac{2}{3}a^3\,d\theta=\frac{2}{3}\left[a^3\theta\right]_0^{2\pi}=\frac{4}{3}\pi a^3$ $\square$



Ex: Compute $\iint_R \sin(\sqrt{x^2+y^2})\, dA$ for R the annulus between the circles $x^2+y^2=1$ and $x^2+y^2=9$.

Sol: Turn the integral polar

$R_{polar} = \{(r, \theta): 0 \le \theta \le 2\pi,\ 1 \le r \le 3\}$

$f(x,y) = \sin(\sqrt{x^2+y^2})$ has polar form

$f(r\cos\theta, r\sin\theta) = \sin(\sqrt{r^2}) = \sin(r)$

$\therefore \iint_{R_{cart}} \sin(\sqrt{x^2+y^2})\, dA_{cart} = \iint_{R_{pol}} \sin(r)\, r\, dA_{pol}$

$$= \int_{\theta=0}^{2\pi} \int_{r=1}^{3} \sin(r) r \, dr\, d\theta \quad \begin{pmatrix} u=r & dv=\sin(r)dr \\ du=dr & v=-\cos(r) \end{pmatrix}$$

$$= \int_{\theta=0}^{2\pi} \left[-r\cos(r) - \int -\cos(r)\, dr\right]_{r=1}^{3} d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[-r\cos(r) + \sin(r)\right]_{r=1}^{3} d\theta = \int_{\theta=0}^{2\pi} ((-3\cos(3)+\sin(3)) - (-1\cos(1)+\sin(1)))$$

$$= \underline{(\sin(3) - \sin(1) - 3\cos(3) + \cos(1))}\, [\theta]_{\theta=0}^{2\pi} = 2\pi(\ \text{''} \quad \text{''}\ )$$

Exercise: compute $\iint_R x\exp(-x^2-y^2)\, dA$ on R the disk of radius R about the origin